JOURNAL OF DIFFERENTIAL EQUATIONS 1, 1-26 (1965)

Comparison Theorems for Nonlinear Second Order Differential Equations

H.-W. KNOBLOCH

Matematisk Institut, Aarhus Universitet, Aarhus, Denmark

1. INTRODUCTION

In this note the second order differential equation

$$x'' = f(t, x, x') \left(' = \frac{d}{dt}\right) \tag{1.1}$$

will be considered and certain statements will be proved about the behavior of the solutions during a fixed, but arbitrary large time interval I. These statements are related and in part generalize the well known classical comparison and oscillation theorems for homogeneous linear equations.

We will first study solutions of (1.1) which are distinguished by a certain property and do not seem to have received attention before. They can be regarded, roughly speaking, as lower (upper) limit of a sequence of neighboring solutions on the whole interval I. The precise definition is the following.

We say that a solution ξ has property (B) if there exists a sequence of solutions ξ_ν such that

(i) $\xi_\nu \to \xi$, $\xi_\nu' \to \xi'$ for $\nu \to \infty$, uniformly for all $t \in I$,

(ii) the difference $\Delta_\nu = \xi - \xi_\nu$ is $\neq 0$ and has the same sign for all ν and all $t \in I$,

(iii) $|\Delta_\nu'| \leqslant c\,|\Delta_\nu|$ for all ν and all $t \in I$ with a constant c, which is independent of ν and t. (1.2)

The purpose of this note is to give both positive and negative criteria for existence of such solutions. It will become clear from the kind of results we are going to obtain that existence or nonexistence depends upon the interval I and reflects a nontrivial global property of the manifold of all solutions.

The following negative criterion is contained in Theorem 1 (Section 3): If ξ is a solution with property (B) then the linear homogeneous equation

$$x'' = f_{x'}(t, \xi, \xi')\, x' + f_x(t, \xi, \xi')\, x$$

(that is the first variation of (1.1) with respect to ξ) can have no solution—except the trivial one—with more than one zero in I. This implies, for example, that an inhomogeneous linear equation can have no solution with property (B), if the corresponding homogeneous equation has a solution $\not\equiv 0$ with more than one zero in I.

On the other hand the existence of a solution with property (B) is a consequence of some surprisingly simple conditions, the essential one being the following:

There exist two sufficiently smooth functions α, β *with* $\alpha < \beta$,

$$-\alpha'' + f(t, \alpha, \alpha') \leqslant 0, \qquad -\beta'' + f(t, \beta, \beta') \geqslant 0 \tag{1.3}$$

for all $t \in I$.

The additional requirements are explicitely stated in Theorem 3, Section 4 and are, for example, all satisfied if $|f|$ does not grow faster than x'^2 for $|x'| \to \infty$. One can then always find a solution ξ with property (B) and

$$\alpha \leqslant \xi \leqslant \beta \tag{1.4}$$

for all $t \in I$.

In view of Theorem 1 it is clear that from the classical Sturm comparison theorems one can easily formulate conditions on f which do not allow the existence of a solution with property (B), and therefore make it impossible for pairs of functions α, β satisfying all hypotheses of Theorem 3 to exist. This leads to interesting consequences to be discussed in Sections 3 and 5. The results will be of this type: If a trajectory starts at $t = a$ in a certain region of the (t, x, x')-space, it has to leave this region not later than $t = b$.

As an example we will consider an autonomous equation of the form

$$x'' = g(x, x') - x \tag{1.5}$$

and prove the following *Theorem 4. Let* ξ *be a solution of (1.5) with* $\xi(t_0) > 0$ *for a certain* t_0. *Put* $\vartheta_0 = \arg(\xi(t_0), \xi'(t_0))$ *with* $|\vartheta_0| < \pi/2$. *Let there be given a number* $\gamma > 0$. *Then the curve* (ξ, ξ') *reaches either the half-plane* $x \leqslant \gamma$ *or the region* $g(x, x') \geqslant \gamma$ *for some* t *with* $t_0 \leqslant t \leqslant t_0 + \vartheta_0 + \pi/2$.

This is actually a special case of a more general theorem concerning an equation of the form $x'' = g(t, x, x') + h(t, x)$.

In the last section we will demonstrate how one can obtain from Theorem 4 concrete and, as it seems, so far unknown information about the global behavior of solutions. The results which we obtain supplement and in part generalize the extensive research done on van der Pol's equation elsewhere [2].

The proof of the two fundamental theorems is based on a certain technique which does not seem to have been used before, so a few remarks about the motivation may be of interest. The idea of the proof stemmed from an observation, made on an earlier occasion [7], when existence theorems for periodic solutions were established on the basis of an analytic method of Cesari (see [4], [6], also [3], chap. 11). It is typical for this method, as used in [7], that it yields under appropriate conditions existence of a periodic solution ξ and at the same time an estimate for ξ and ξ'. The estimates for ξ are of the form (1.3), (1.4) with α, β being periodic over the interval I. Now one may ask whether it is possible to relax the periodicity conditions for α, β and still preserve the inequalities (1.4) (but of course not the periodicity of ξ). A natural way to attempt such a generalization is to make a suitable extension of α, β and of the right hand side of (1.1) into some bigger t-interval, for which Cesari's method then will work. This is in short the basic idea behind the considerations of the following two sections.

2. Preliminary Remarks. Lemmas

In this section we mostly review material from [7] and other sources, which will be needed later.

The notation will follow closely the one adopted in [5] (see especially chap. 1), with few exceptions to be explained below. Intervals on the t-axis will be denoted by $[a, b]$ or I. Regions of the (t, x)-plane (the (t, x, x')-space) are subregions of $I \times R$ ($I \times R^2$) for a suitable I and are usually denoted by D. The symbols ω, Ω, however, always refer to the special type of regions described by (3.2) of [7], p. 183, that is

$$\begin{aligned}\omega &= \{(t, x) : t \in I, \alpha(t) \leqslant x \leqslant \beta(t)\}\\ \Omega &= \{(t, x, x') : (t, x) \in \omega\}\end{aligned} \tag{2.1}$$

It is convenient for our purposes to work with function classes $C^1(D)$, $C_p{}^1(D)$ also for closed regions D. The definition used will then be the following.

$f \in C^k(D)$ means: There exists an open region D^* containing D and a function $f^* \in C^k(D^*)$ such that $f = f^*$ on D.

$f \in C_p{}^k(D)$ means: (i) $f \in C^{k-1}(D)$ provided $k \geqslant 1$, (ii) D can be decomposed into finitely many closed regions D_i such that $f \mid D_i$ is of class $C^k(D_i)$ for all i.

For those functions f which appear on the right hand side of differential equations we will always make the following smoothness requirements in some region $D \subseteq I \times R^2$:

The interval I can be decomposed into finitely many subintervals $[t_i, t_{i+1}]$ such that

(i) $f \in C(D_i)$, where D_i is the intersection of D with the strip $t_i \leqslant t \leqslant t_{i+1}$,

(ii) f satisfies a *local* Lipschitz-condition with respect to x, x' in each D_i.

This will be written in the form

$$f \in (C_t\,, \text{lip}) \text{ on } D. \tag{2.2}$$

It should be noted that if (2.2) holds and if $(t, \xi, \xi') \in D$ for all $t \in I$ with some function $\xi \in C_p^2(I)$, then

$$F = -\xi'' + f(t, \xi, \xi')$$

is of class $C_p(I)$. So there can be no doubt about the meaning of a statement like this: $F \leqslant 0$ for all $t \in I$.

A function $\alpha = \alpha(t)$ will be called a *lower* solution over the interval I of the differential equation

$$x'' = f(t, x, x')$$

if

$$\alpha \in C_p^2(I) \qquad \text{and} \qquad -\alpha'' + f(t, \alpha, \alpha') \leqslant 0 \quad \text{for all} \quad t \in I.$$

Similarly we define *upper* solutions β by the properties

$$\beta \in C_p^2(I) \qquad \text{and} \qquad -\beta'' + f(t, \beta, \beta') \geqslant 0 \quad \text{for all} \quad t \in I.$$

A *solution* ξ is a function which is upper and lower solution at the same time. Strictly speaking, ξ is then a solution in the extended sense (see [5], Chap. 2, Section 1.2), but the extension is so slight that the usual uniqueness theorem obviously holds.

LEMMA 1. *Let $f \in (C_t\,, \text{lip})$ on $D \subseteq I \times R^2$ and let ξ_ν be a sequence of solutions of the differential equation*

$$x'' = f(t, x, x') \tag{2.3}$$

with $(t, \xi_\nu\,, \xi_\nu') \subseteq D$ for all $t \in I$. Assume that the sequences $\{\xi_\nu\}$ and $\{\xi_\nu'\}$ converge uniformly on I. Then $\xi = \lim \xi_\nu$ is also a solution of (2.3).

Proof: Let $t = t_j$ be the finitely many planes in the (t, x, x')-space, where f has discontinuities. Since ξ_ν is an exact solution, ξ_ν'' exists and is continuous on every interval $[t_j\,, t_{j+1}]$ and

$$\xi_\nu'' = f(t, \xi_\nu\,, \xi_\nu').$$

Now if t is restricted to $[t_j\,, t_{j+1}]$, f becomes a continuous function in all variables and this implies that ξ_ν'' converges uniformly with limit ξ'' on $[t_j\,, t_{j+1}]$ and we have

$$\xi'' = f(t, \xi, \xi').$$

We are now ready to prove several lemmas, which provide all the background needed later.

LEMMA 2. *Let α be a lower, β an upper solution of (2.3) on the interval $I = [a, b]$ and let $f \in (C_t, \text{lip})$ in Ω (see (2.1)). Let be $\alpha \leqslant \beta$ for all $t \in I$, $\alpha < \beta$ for $t = a, b$. Then we have $\alpha < \beta$ for all $t \in I$.*

Proof: Without loss of generality we may assume that $|f|$ is bounded on Ω, because the values of f outside a finite region containing the curves (t, α, α') and (t, β, β') are involved only in so far as the property $f \in (C_t, \text{lip})$ is concerned. So we can always make $|f|$ bounded by changing the function outside a certain finite region. Furthermore, all hypotheses of the lemma hold if we replace f by

$$f_1 = f(t, x, x') + \beta'' - f(t, \beta, \beta').$$

We have then

$$\beta'' = f_1(t, \beta, \beta'),$$
$$-\alpha'' + f_1(t, \alpha, \alpha') = -\alpha'' + f(t, \alpha, \alpha') - (-\beta'' + f(t, \beta, \beta')) \leqslant 0.$$

Finally we can reduce β to 0 by means of the substitution

$$f_1 \to f_1(t, x + \beta, x' + \beta') - \beta''.$$

So for the proof we may assume that one has to deal with the following situation:

$$\begin{gathered} f(t, 0, 0) = 0, \qquad -\alpha'' + f(t, \alpha, \alpha') \leqslant 0, \\ \alpha \leqslant 0 \quad \text{for all} \quad t \in I, \qquad \alpha < 0 \quad \text{for} \quad t = a, b. \\ |f| \quad \text{bounded in} \quad \Omega. \end{gathered} \tag{2.4}$$

We choose a number $p > 0$ and extend the definition of α into the range $t > b$ such that

$$\begin{gathered} \alpha \text{ is of class } C^1 \qquad \text{for} \qquad t \geqslant a, \\ \alpha'' = p, \; t > b. \end{gathered} \tag{2.5}$$

Because of $\alpha(b) < 0$ we can certainly find a $b' > b$ such that

$$\alpha < 0 \qquad \text{for} \qquad t \in [b, b'].$$

We then choose a number $a' < a$ and extend the definition of α into the interval $[a', a]$ such that

$$\begin{gathered} \alpha \in C^1([a', b']), \\ \alpha < 0 \qquad \text{if} \qquad t \in [a', a], \\ \alpha(a') = \alpha(b'), \qquad \alpha'(a') = \alpha'(b'). \end{gathered} \tag{2.6}$$

Since $\alpha(a) < 0$, $\alpha(b') < 0$ these three conditions are compatible and the extension is therefore always possible.

Let now $f_1 = f_1(t, x, x')$ be defined as

$$f_1 = \begin{cases} kx & \text{for} \quad t \in [a', a) \\ f & \text{for} \quad t \in [a, b] \\ p & \text{for} \quad t \in (b, b'], \end{cases} \tag{2.7}$$

where $k > 0$ is a sufficiently large constant to be determined later. Clearly:

$$\begin{aligned} &f_1 \in (C_t, \mathrm{lip}) \text{ in } \Omega_1 = \{(t, x, x') : t \in [a', b'], \alpha \leqslant x \leqslant \beta\} \\ &| f_1 | \text{ is bounded in } \Omega_1 . \end{aligned} \tag{2.8}$$

It also follows immediately from (2.7) that $\beta = 0$ is an upper solution of

$$x'' = f_1(t, x, x') \tag{2.9}$$

on the interval $I_1 = [a', b']$ and that it is an exact solution on the subinterval $[a', b]$.

Furthermore, we have $-\alpha'' + f_1(t, \alpha, \alpha') \leqslant 0$ certainly for $a \leqslant t \leqslant b'$, according to (2.4) and (2.5), and

$$-\alpha'' + f_1(t, \alpha, \alpha') = -\alpha'' + k\alpha \qquad \text{for} \qquad a' \leqslant t \leqslant a.$$

Since $\alpha < 0$ on $[a', a]$ the last expression can be made $\leqslant 0$ by taking k sufficiently large, α is then a lower solution on the whole interval I_1.

The differential equation (2.9) and the extended functions $\alpha, \beta (= 0)$ therefore satisfy all hypotheses of Theorem 2 of [7] (p. 188), with Ω_1 instead of Ω. The periodicity of α is guaranteed by (2.6).

Hence there exists a periodic solution ξ of the differential equation (2.9) for which

$$\alpha \leqslant \xi \leqslant 0 \tag{2.10}$$

holds for all $t \in I_1$.

It follows now from (2.10) that if $\xi(t_0) = 0$ for $a' < t_0 < b'$, then we have also $\xi'(t_0) = 0$. Since $\beta = 0$ is an exact solution over $[a', b]$ the relations $\xi(t_0) = 0$, $\xi'(t_0) = 0$, with $a' < t_0 < b$ imply $\xi(t) = 0$ for all $t \in [a', b]$, especially

$$\xi(b) = 0, \qquad \xi'(b) = 0.$$

But we have from (2.7), $\xi'' = p > 0$ for $t > b$, and so the last equalities would lead to $\xi > 0$ for all $t > b$ which is impossible in view of (2.10).

Therefore we must have $\xi < 0$ for all $t \in (a', b)$ and hence also $\alpha < 0$ for $t \in (a, b)$. This was to be proved.

LEMMA 3. *Let α be a lower, $\beta \geqslant \alpha$ an upper solution of (2.3) on an interval I and let $f \in (C_t, \mathrm{lip})$ in Ω. Let α, β be periodic with respect to I (that means,*

α, α', β, β' assume the same values at the end points of I). Then we have either

$$\alpha < \beta \qquad \textit{for all} \qquad t \in I$$

or $\alpha = \beta$ everywhere and α, β are exact solutions.

Proof. Let α, β, and f, by periodicity, be defined for all t with $-\infty < t < \infty$. α is then a lower, $\beta \geqslant \alpha$ an upper solution and both are of class C_p^2 for all t. Assume now that there exist points t_0, t_1 such that

$$\alpha(t_0) = \beta(t_0), \qquad \alpha(t_1) < \beta(t_1).$$

Since α, β are periodic with the same period, we can then find an interval $[a, b]$ such that

$$a < t_0 < b,$$
$$\alpha < \beta \qquad \text{for} \qquad t = a, b.$$

But then we would have $\alpha < \beta$ also for $t = t_0$ according to Lemma 2.

LEMMA 4. *Let the hypotheses for α, β, f be the same as in Lemma 3 with the following addition*

(i) $|f| \leqslant C\,|x'|^2$ *for* $|x'| \to \infty$ *with some constant* C.

(ii) *neither α nor β is an exact solution over I.*

Then the differential equation

$$x'' = f(t, x, x')$$

has a solution ξ, which is periodic with respect to I and satisfies

$$\alpha < \xi < \beta \qquad \textit{for all} \qquad t \in I.$$

Proof: That there exists a periodic solution ξ with

$$\alpha \leqslant \xi \leqslant \beta$$

follows again from [7], Theorem 2. Since α, β are not exact solutions, we cannot have $\alpha = \xi$ or $\beta = \xi$ for all $t \in I$ and so we must have strict inequality for all t, according to Lemma 3.

LEMMA 5. *Given an interval $I = [a, b]$ and regions ω, Ω of the form (2.1). Let $\theta(t, x)$ and $f(t, x, x')$ be functions satisfying the following conditions*

(i) $\theta \in C_p^1(\omega)$, $f \in (C_t, \text{lip})$ *in* Ω,

(ii) $\theta(a, x) = \theta(b, x)$ *for all* x,

(iii) $f(t, x, \theta) - \theta_x\theta - \theta_t \neq 0$ *for all* $(t, x) \in \omega$.[1]

[1] At points of discontinuity all limits should be $\neq 0$ and have the same sign.

Furthermore let there exist an integrable function ϑ on I such that either one of the conditions (iv), (iv') holds:

$$\text{(iv)}\ \theta(t, x) \leqslant \vartheta(t) \quad \text{for all} \quad (t, x) \in \omega, \quad \int_a^b \vartheta dt \leqslant 0,$$

$$\text{(iv')}\ \theta(t, x) \geqslant \vartheta(t) \quad \text{for all} \quad (t, x) \in \omega, \quad \int_a^b \vartheta dt \geqslant 0.$$

If ξ is then a solution of

$$x'' = f(t, x, x'),$$

periodic with respect to I, and if $(t, \xi) \in \omega$ for all $t \in I$ we have

$$\xi' > \theta(t, \xi) \text{ in case (iv)}, \qquad \xi' < \theta(t, \xi) \text{ in case (iv')}$$

for all $t \in I$.

Proof: Is the same as for Lemma 7,8 in [7], p. 189, if one replaces ω' by ω. It should be observed that the expression occuring in (iii) is the inner product of the upward normal to the surface $y = \theta(t, x)$ with a vector tangent to the trajectory through $(t, x, \theta(t, x))$, the latter being oriented by increasing t. Therefore:

$$f(t, x, \theta) - \theta_x\theta - \theta_t > 0\ (< 0) \text{ means:}$$

The trajectories cross the surface $y = \theta$ upwards (downwards) with increasing t. (2.11)

LEMMA 6. *Let $f \in (C_t, \text{lip})$ in $I \times R^2$ and let D be a bounded subregion. Let there be given a number $\delta > 0$. Then there exist constants k_0, k_1, depending on f, D, δ only, such that the following is true. Whenever we have two solutions ξ, η of the differential equation*

$$x'' = f(t, x, x')$$

with

$$(t, \xi, \xi') \in D, \qquad (t, \eta, \eta') \in D \quad \text{for all} \quad t \in I$$

and

$$| \xi - \eta | \leqslant \delta, \qquad | \xi' - \eta' | \leqslant \delta$$

for some $t_0 \in I$, then

$$| \xi' - \eta' | \leqslant k_0 | \xi(t_0) - \eta(t_0) | + k_1 | \xi'(t_0) - \eta'(t_0) |$$

for all $t \in I$.

Proof: We use, for the moment, vector notation and put

$$\Delta = (\xi(t_0) - \eta(t_0), \xi'(t_0) - \eta'(t_0)),$$
$$\boldsymbol{\xi} = (\xi, \xi'), \qquad \boldsymbol{\eta} = \boldsymbol{\xi} + \Delta.$$

$\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ then have the same initial values at $t = t_0$ and satisfy differential equations of the form

$$\mathbf{x}' = X(t, \mathbf{x}) \qquad \mathbf{x}' = X(t, \mathbf{x} - \Delta) \tag{2.12}$$

respectively.

Let now $D(\delta)$ be the set of all points of $I \times R^2$, which have distance $\leqslant \delta\sqrt{2}$ from D. We have then $D \subset D(\delta)$ and $(t, \mathbf{x} \pm \Delta) \in D(\delta)$ if $(t, \mathbf{x}) \in D$. In particular $(t, \boldsymbol{\xi})$, $(t, \boldsymbol{\eta}) \in D(\delta)$ for all $t \in I$. Since $D(\delta)$ is bounded, X satisfies a global Lipschitz-condition in $D(\delta)$ and there is therefore a constant k, depending only on f and $D(\delta)$, such that

$$| X(t, \mathbf{x} - \Delta) - X(t, \mathbf{x}) | \leqslant k | \Delta |$$

for all $(t, \mathbf{x}) \in D$ ($| \cdots |$ denotes maximum norm). In other words, the right hand sides of the two differential equations (2.12) differ by a quantity $\leqslant k | \Delta |$ in absolute value. Everything follows now from a standard result (see, e.g., [1], Theorem (1.6.II) p. 55. It is actually required there that X is of class C, but the proof covers our case also).

LEMMA 7. *Let $a' < a < b < b'$ and let φ be a differentiable function of t on (a', b'). Let φ' have at least one zero in each of the intervals $(a', a]$, $[b, b')$. Then the minimum of φ on $[a, b]$ is not smaller than the minimum of φ on the set $\{t : t \in [a', b'], \varphi'(t) = 0\}$.*

Proof. The lemma is certainly true if $\varphi'(a) = 0$, $\varphi'(b) = 0$. It follows from the hypotheses of the lemma, that there exists an interval $[a'', b'']$ such that $\varphi'(a'') = 0$, $\varphi'(b'') = 0$ and $[a, b] \subseteq [a'', b'']$. Now the minimum of φ on $[a, b]$ is not smaller than the minimum of φ on $[a'', b'']$.

LEMMA 8. *Let S be a closed bounded set in (t, x, x')-space and $\epsilon > 0$, $K > 0$ be given numbers. Then there exists a function f of class C^1 in the whole space such that $f \geqslant K$ on S and $f = 0$ for all points which have distance $> \epsilon$ from S.*

Proof: By standard technique.

LEMMA 9. *Given functions $\alpha_j(t)$, $\tau_j(t) \in C([a, b])$ $(j = 1, \cdots, N)$ and functions $f_i(x)$, $g_i(x)$, $i = 0, 1, \cdots$ which are bounded and of class C^1 in*

$-\infty < x < \infty$. *Assume that the following relations hold* (*with* $t_0 = a$, $t_1 = b$).

$$f_i < g_i \quad \textit{for all} \quad x \qquad \textit{and} \qquad i = 0, 1, \tag{2.13}$$

$$f_i(\alpha_j(t_i)) < \tau_j(t_i) < g_i(\alpha_j(t_i)) \quad \textit{for} \quad i = 0, 1, \qquad \textit{and} \qquad j = 1, \cdots, N. \tag{2.14}$$

Let there be given a number $K > 0$. *Then there exist functions* $\phi(t, x)$, $\psi(t, x)$ *of class* C^1 *on* $[a, b] \times R$ *and integrable functions* $\varphi(t)$, $\chi(t)$ *on* $[a, b]$ *which have the following properties*

$$\phi < \psi \textit{ everywhere},$$

$$\phi(t_i, x) = f_i(x), \qquad \psi(t_i, x) = g_i(x) \qquad \textit{for} \qquad i = 0, 1, \tag{2.15}$$

$$\phi(t, \alpha_j(t)) < \tau_j(t) < \psi(t, \alpha_j(t)) \quad \textit{for all} \quad t \in [a, b] \qquad \textit{and} \qquad j = 1, \cdots, N, \tag{2.16}$$

$$\phi(t, x) \leqslant \varphi(t), \qquad \int_a^b \varphi dt \leqslant -K, \qquad \psi(t, x) \geqslant \chi(t),$$

$$\int_a^b \chi dt \geqslant K \qquad \textit{for all} \qquad (t, x) \in [a, b] \times R. \tag{2.17}$$

Proof: We first choose two functions $\rho_i(t) \in C^1([a, b])$, $(i = 0, 1)$, such that

$$\rho_i \geqslant 0 \qquad \text{for} \qquad t \in [a, b], \qquad \rho_0(a) = 1, \qquad \rho_0(b) = 0,$$
$$\rho_1(a) = 0, \qquad \rho_1(b) = 1.$$

Then we put

$$\phi = f_0(x)\,\rho_0(t) + f_1(x)\,\rho_1(t) - \kappa(t)$$
$$\psi = g_0(x)\,\rho_0(t) + g_1(x)\,\rho_1(t) + \kappa(t)$$

with some $\kappa \in C^1([a, b])$ to be specified below.

To satisfy (2.15) we must have

$$\kappa = 0 \quad \text{for} \quad t = a, b, \qquad \kappa > 0 \quad \text{for} \quad a < t < b. \tag{2.18}$$

Condition (2.16) leads to finitely many inequalities of the form

$$\kappa > \lambda_j \qquad \text{for all} \qquad t \in [a, b]. \tag{2.19}$$

Here $\lambda_j \in C([a, b])$ and $\lambda_j < 0$ for $t = a, b$, because of (2.14). So (2.19) does not interfere with (2.18).

Since f_i, g_i are supposed to be bounded we have estimates of the form

$$\phi \leqslant -\kappa + c_1, \qquad \psi \geqslant \kappa + c_2$$

with certain constants c_i which are independent from κ. Hence an inequality of the form

$$\int_a^b \kappa dt \geqslant K_1, \qquad \text{with a suitable constant} \qquad K_1, \tag{2.20}$$

will certainly be sufficient to guarantee (2.17).

So all that is needed is a κ satisfying (2.18)-(2.20) and such a one obviously exists.

3. Solutions with Property (B) and the Equation of First Variation

In this section we are going to investigate the first variation of a differential equation with respect to a solution which has property (B).

Lemma 10. *Let there be given, on some interval $I = [a, b]$, functions $l(t)$, $m(t)$, $\xi(t)$ with the following properties*

(i) $l, m \in C_p(I)$, $\xi \in C_p^2(I)$,

(ii) *the two functions* ξ, $\xi'' + l(t)\,\xi' + m(t)\,\xi$ *have opposite constant sign in* I,[2]

(iii) $\xi(a) \neq 0$, $\xi(b) \neq 0$.

Then the linear homogeneous equation

$$x'' + lx' + mx = 0 \tag{3.1}$$

has a solution without zeros on I.

Proof. We may assume without loss of generality that $\xi \leqslant 0$, $\xi'' + l\xi' + m\xi \geqslant 0$. ξ is then a lower solution of (3.1) on I and $\xi(a)$, $\xi(b)$ are < 0 because of (iii). Obviously $\beta = 0$ is an upper solution. We now extend the definition of $\alpha = \xi$, $\beta = 0$, $f = -lx' - mx$ to some bigger interval $[a', b'] = I_1$ in the same way as it was done in the proof of Lemma 2. Then α, β become periodic upper and lower solutions with respect to I_1, but not exact solutions everywhere. It follows immediately from Lemma 4—applied to the extended differential equation—that there exists a solution of (3.1) which is negative for all $t \in [a, b]$.

Remark. If l is continuous on whole I, one can reduce (3.1) to the form $x'' + mx = 0$ and then prove the lemma with the standard technique used in comparison theorems.

[2] That means: One is > 0, the other < 0 for all $t \in I$.

THEOREM 1. *Let ξ be a solution with property (B) of a differential equation*

$$x'' = f(t, x, x') \tag{3.2}$$

on some interval I. Assume that there exists a region D such that $(t, \xi, \xi') \in D$ for all $t \in I$ and

$$f, f_x, f_{x'} \in C_t \text{ on } D.$$

Let μ be a function of class C_p on I with

$$f_x(t, \xi, \xi') < \mu \tag{3.3}$$

for all $t \in I$, including the limits at points of discontinuity. Then the linear homogeneous equation

$$x'' = f_{x'}(t, \xi, \xi')\, x' + \mu x \tag{3.4}$$

has a solution without zeros on I.

Proof: Let ξ_ν, ξ_ν' be a sequence of solutions and their derivatives which has the properties (1.2) and converges uniformly to ξ and ξ' respectively.

We put

$$\Delta_\nu = \xi - \xi_\nu$$

and have then

$$\begin{aligned} \Delta_\nu'' &= \xi'' - \xi_\nu'' = f(t, \xi, \xi') - f(t, \xi_\nu, \xi_\nu') \\ &= f_{x'}(t, \xi, \xi')\, \Delta_\nu' + f_x(t, \xi, \xi')\, \Delta_\nu + O(|\Delta_\nu| + |\Delta_\nu'|). \end{aligned}$$

But $|\Delta_\nu'| = o(|\Delta_\nu|)$ in view of (1.2) (iii) and hence

$$\Delta_\nu'' - f_{x'}(t, \xi, \xi')\, \Delta_\nu' - \mu(t)\, \Delta_\nu = (f_x(t, \xi, \xi') - \mu(t))\, \Delta_\nu + O(|\Delta_\nu|).$$

It follows from (3.3) that if $|\Delta_\nu|$ is sufficiently small the expression

$$\Delta_\nu'' - f_x(t, \xi, \xi')\, \Delta_\nu' - \mu(t)\, \Delta_\nu$$

has the opposite sign to Δ_ν and the latter does not change on I, according to (1.2).

The theorem follows now immediately from Lemma 10, applied to (3.4) with $\xi = \Delta_\nu$.

Remark. If we have only $f_x(t, \xi, \xi') \leqslant \mu$ on I (instead of (3.3)) then every nontrivial solution of (3.4) has at most one zero in the interior of I. Otherwise there would exist a nontrivial solution of (3.4) with more than one zero in the interior of I and the same would still be true if we replace μ by $\mu + \epsilon$, where ϵ is > 0 and sufficiently small. It follows then from the separation theorem that every solution of

$$x'' = f_{x'}(t, \xi, \xi')\, x' + (\mu + \epsilon)\, x$$

has a zero on I in contradiction to what we have proved before.

As an application we can now prove the following theorem for autonomous equations.

THEOREM 2. *Let there be given a differential equation*

$$x'' = f(x, x'), \tag{3.5}$$

where f does not depend upon t and is of class C^1 in some region D of the (x, x')-plane. Let ξ be a solution of (3.5) on some interval I with $(\xi, \xi') \in D$ and $\xi' \neq 0$ for all $t \in I$. Then every nontrivial solution of the equation of first variation of f

$$x'' = f_{x'}(\xi, \xi')\, x' + f_x(\xi, \xi')\, x$$

has at most one zero in the interior of I.

Proof: We simply have to show, that ξ has property (B) with respect to I. Let us assume that $\xi' > 0$ on I (the other case can be treated in an analogous way). This implies that we have $\xi(t + \delta) > \xi(t)$ for all $t \in I$ provided that δ is > 0 and sufficiently small. Since ξ' is bounded away from zero on I we can find a constant $k_1 > 0$, independent of t and δ, such that

$$| \xi(t + \delta) - \xi(t) | \geqslant k_1 \delta.$$

Since $\xi'' = f(\xi, \xi')$ is continuous on I there is also a constant k_2 such that

$$| \xi'(t + \delta) - \xi'(t) | \leqslant k_2 \delta.$$

So a sequence ξ_ν with the desired properties is easy to construct: Put

$$\xi_\nu = \xi(t + \nu^{-1}).$$

4. THE GENERAL EXISTENCE THEOREM

We are now going to establish more general conditions which guarantee the existence of solutions with property (B).

LEMMA 11. *Let there be given a differential equation*

$$x'' = f(t, x, x'), \tag{4.1}$$

a lower solution α *and an upper solution* β *with*

$$\alpha < \beta$$

on some interval I. Let ω, Ω *be defined as in (2.1) and let* $f \in (C_t, \text{lip})$ *in* Ω.
Furthermore, assume that there are two functions $\phi, \psi \in C_p^1(\omega)$ *such that*

(i) $f(t, x, \theta) - \theta_x \theta - \theta_t$ *is* $\neq 0$ *and has constant sign (including the limits at points of discontinuity) on* ω, *for* $\theta = \phi, \psi$,

(ii) $\phi < \psi$ *for all* x *and* $t = a, b$,

$$\phi(t, \alpha) < \alpha' < \psi(t, \alpha), \qquad \phi(t, \beta) < \beta' < \psi(t, \beta) \qquad \textit{for} \quad t = a, b. \tag{4.2}$$

Then the differential equation (4.1) has a solution ξ *satisfying*

$$\alpha < \xi < \beta$$
$$\phi(t, \xi) < \xi' < \psi(t, \xi) \tag{4.3}$$

for all $t \in I$.

Proof. We may assume that $|f|$ is bounded in Ω. That this means no loss in generality follows by the same argument as in the proof of Lemma 2. This time, however, the subregion has to be taken so large that it contains not only the curves (t, α, α'), (t, β, β'), but also the surfaces $y = \phi, \psi$.

We choose now a number $b' > b$ and extend the definition of α, β into $[b, b']$ such that

(i) $\alpha, \beta \in C_p^2([a, b'])$,
(ii) $\alpha(a) = \alpha(b'), \quad \beta(a) = \beta(b'), \quad \alpha'(a) = \alpha'(b'), \quad \beta'(a) = \beta'(b')$,
(iii) $\alpha < \beta$ on $[a, b']$. (4.4)

Let ω_1, Ω_1 then denote the regions defined by (2.1) for $I = I_1 = [a, b']$.

Next we wish to extend the definition of ϕ, ψ into the region ω_1. The following conditions shall hold:

(i) $\phi, \psi \in C_p^1(\omega_1)$
(ii) $\phi(a, x) = \phi(b', x), \quad \psi(a, x) = \psi(b', x)$ for all x,
(iii) $\phi < \psi$ for $t \geqslant b$,
(iv) $\phi(t, \alpha) < \alpha' < \psi(t, \alpha)$, $\phi(t, \beta) < \beta' < \psi(t, \beta)$ for $t \geqslant b$,
(v) ϕ, ψ can be estimated by integrable functions φ, χ in the form

$$\phi(t, x) \leqslant \varphi(t), \qquad \int_a^{b'} \varphi dt \leqslant 0, \qquad \psi(t, x) \geqslant \chi(t),$$
$$\int_a^{b'} \chi dt \geqslant 0 \qquad \text{for all} \qquad (t, x) \in \omega_1 . \tag{4.5}$$

Part (v) can easily be modified such that only those (t, x) are involved for which $t \geqslant b$. Let the constants c_1, c_2 be determined such that

$$\phi \leqslant c_1, \qquad \psi \geqslant c_2 \qquad \text{for all} \qquad (t, x) \in \omega.$$

(v) is then certainly true if we have

(v′) $\phi \leqslant \varphi, \qquad \psi \geqslant \chi \qquad \text{for} \qquad t \geqslant b \qquad \text{and}$

$$\int_b^{b'} \varphi dt \leqslant -c_1(b-a), \qquad \int_b^{b'} \chi dt \geqslant c_2(b-a).$$

The construction of ϕ, ψ in the range $t > b$ becomes a problem of the kind we have discussed in Lemma 9 if we make the identifications

$$\begin{aligned} & f_0 = \phi(b, x), \qquad g_0 = \psi(b, x), \\ & f_1 = \phi(a, x), \qquad g_1 = \psi(a, x), \\ N = 2, \qquad & \alpha_1 = \alpha, \qquad \tau_1 = \alpha', \qquad \alpha_2 = \beta, \qquad \tau_2 = \beta', \\ & K = \text{Max}\,(|\,c_1\,|, |\,c_2\,|)\,(b-a). \end{aligned}$$

That all hypotheses of Lemma 9 hold can be seen immediately from (4.2) (ii). So the extension of ϕ, ψ can be performed in the desired way.

It follows from (4.5), (iii), (iv) that the surfaces $y = \phi$, ψ have no points in common with each other and with the curves (t, α, α'), (t, β, β') for $t \geqslant b$. Also the latter do not intersect each other because of $\alpha < \beta$. So, for $b \leqslant t \leqslant b'$, the surfaces and the curves represent four closed bounded and mutually distinct point sets in (t, x, x')-space. Therefore, using Lemma 8, we can find a function $f_1 = f_1(t, x, x')$ with these properties:

(i) f_1 is bounded and of class C^1 in $[b, b'] \times R^2$,

(ii) $-\alpha'' + f_1(t, \alpha, \alpha') < 0, \qquad -\beta'' + f_1(t, \beta, \beta') > 0$ for $b \leqslant t \leqslant b'$,

(iii) $f_1(t, x, \theta) - \theta_x\theta - \theta_t \; (\theta = \phi, \psi)$ is $\neq 0$ for all $(t, x) \in \omega_1$ with $b \leqslant t \leqslant b'$ and has the same sign as $f(t, x, \theta) - \theta_x\theta - \theta_t$ in the range $t < b$. (4.6)

Let now $\tilde{f} = \tilde{f}(t, x, x')$ be defined by

$$\tilde{f} = \begin{cases} f & \text{for} \quad a \leqslant t \leqslant b, \\ f_1 & \text{for} \quad b \leqslant t \leqslant b'. \end{cases}$$

$|\tilde{f}|$ is bounded and $\tilde{f} \in (C_t, \text{lip})$ in Ω_1. It follows from the hypotheses of our lemma and from (4.6), (ii) that α is a lower, β an upper solution of

$$x'' = \tilde{f}(t, x, x') \tag{4.7}$$

on the interval $[a, b']$ and none of them is an exact solution. According to Lemma 4 the differential equation (4.7) admits then a solution ξ which is periodic over $[a, b']$ and satisfies

$$\alpha < \xi < \beta$$

for all $t \in [a, b']$. Furthermore, we see from (4.2) (i), (4.5) (v) and (4.6) (iii) that ξ and the functions $\theta = \phi, \psi$ satisfy all hypotheses of Lemma 5 and therefore we have the inequalities

$$\phi(t, \xi) < \xi' < \psi(t, \xi)$$

for all $t \in [a, b']$.

If t is restricted to $[a, b]$, ξ becomes a solution of the original equation and $\alpha, \beta, \phi, \psi$ have the original meaning. Hence the lemma is proved.

COROLLARY. *Let all assumptions of Lemma 11 hold. Then there exists a sequence of solutions ξ_ν of the differential equation (4.1) such that*

(i) *each ξ_ν satisfies (4.3) for all $t \in I$,*

(ii) *ξ_ν, ξ_ν' converge uniformly on I,*

(iii) *$\xi_{\nu+1} < \xi_\nu$ for all ν and all $t \in I$.*

There also exists a sequence of solutions such that (i), (ii), and

(iii′) *$\xi_{\nu+1} > \xi_\nu$*

hold for all ν and all $t \in I$.

Proof: We know from the lemma that there exists a solution $\xi = \xi_1$ which satisfies the conditions (4.3). It follows from these inequalities that Lemma 11 can be applied to the pair $\alpha, \beta = \xi_1$ as well as to the pair $\alpha = \xi_1$, β and the same f, ϕ, ψ. So we obtain a solution ξ_2 with

$$\alpha < \xi_2 < \xi_1 < \beta \text{ in the first case,}$$
$$\alpha < \xi_1 < \xi_2 < \beta \text{ in the second case}$$

and

$$\phi(t, \xi_2) < \xi_2' < \psi(t, \xi_2).$$

This procedure can be continued and yields a sequence ξ_ν of solutions with properties (i) and (iii) (or (iii′)). It follows now from (4.3) that the trajectories (t, ξ_ν, ξ_ν') are contained in a bounded subregion of Ω, for all ν and all $t \in I$. Therefore $|\xi_\nu|$, $|\xi_\nu'|$, and $|\xi_\nu''| = |f(t, \xi_\nu, \xi_\nu')|$ are bounded uniformly in ν and t. By a well known argument one can then select a subsequence which converges uniformly and the same is true for the sequence of the derivatives.

THEOREM 3. *Let all hypotheses of Lemma 11 hold. Then there exists a solution ξ of the differential equation*

$$x'' = f(t, x, x')$$

which has property (B) and besides satisfies the inequalities

$$\alpha \leqslant \xi \leqslant \beta, \qquad \phi(t, \xi) \leqslant \xi' \leqslant \psi(t, \xi) \tag{4.8}$$

for all $t \in I$.

Proof: Let ξ be the limit of the sequence $\{\xi_\nu\}$ with $\xi_{\nu+1} < \xi_\nu$ which we have constructed above. ξ is a solution of the differential equation (according to Lemma 1) and satisfies (4.8) as well as (1.2), (i) and (ii), but not necessarily (iii). To find a sequence $\{\xi_\nu\}$ which has also this crucial property we apply the results of Lemma 11 and the corollary not directly to the given differential equation, but first perform several extension procedures similar to the one used in the proof of Lemma 11.

Step 1. We choose a number $b' > b$ and extend α, β into the interval $[a, b']$ such that

(i) $\alpha, \beta \in C_p{}^2([a, b'])$, $\alpha < \beta$ for all $t \in [a, b']$,
(ii) $\beta(b') = 0$, $\beta'(b') = 0$, $\alpha'(b') < 0$. (4.9)

ω_1, Ω_1 will then have the usual meaning with $I = I_1$ being the interval $[a, b']$. We put

$$r = + \sqrt{\alpha(b')^2 + \alpha'(b')^2} \tag{4.10}$$

and determine two functions $\phi_0(x)$, $\psi_0(x)$ such that

(i) ϕ_0, ψ_0 are of class C^1 on the interval $-r \leqslant x \leqslant 0$,
(ii) $\phi_0 > r^2$, $\psi_0 > r^2$ for all $x \in [-r, 0]$,
(iii) $-x - \frac{1}{2}(\phi_0)_x$ and $-x - \frac{1}{2}(\psi_0)_x$ are both $\neq 0$ on the interval $[-r, 0]$ and have the same sign as $f(t, x, \phi) - \phi_x\phi - \phi_t$ and $f(t, x, \psi) - \psi_x\psi - \psi_t$ respectively. (4.11)

Functions with these properties obviously exist. Next we extend the definition of ϕ, ψ into the region ω_1. These conditions shall hold:

(i) $\phi, \psi \in C_p{}^1(\omega_1)$,
(ii) $\phi(b', x) = -\sqrt{\phi_0}$, $\psi(b', x) = +\sqrt{\psi_0}$ for $\alpha(b') \leqslant x \leqslant 0$,
(iii) $\phi < \psi$, $\phi(t, \alpha) < \alpha' < \psi(t, \alpha)$, $\phi(t, \beta) < \beta' < \psi(t, \beta)$ for all (t, x) with $b' < t \leqslant b$. (4.12)

According to Lemma 9 such an extension is always possible provided that (ii) and (iii) do not interfere at the boundary $t = b, b'$. Compatibility of (ii) and (iii) at $t = b$ is one of the hypotheses of the theorem, namely, (4.2) (ii). For $t = b'$ we know from (4.9), (4.10) and (4.11) (ii) that

$$\begin{gathered} |\alpha'| \leqslant r, \qquad \beta' = 0, \qquad -\sqrt{\phi_0} < -r, \\ \sqrt{\psi_0} > r \qquad \text{for} \qquad -r \leqslant x \leqslant 0 \end{gathered} \tag{4.13}$$

and

$$-r \leqslant \alpha(b') < 0.$$

So if $t = b'$, (iii) becomes a consequence of (ii).

Now condition (iii) allows the same geometric interpretation as in the proof of Lemma 11: The surfaces $y = \phi, \psi$ and the curves (t, α, α'), (t, β, β') are closed bounded and mutually distinct point sets. So we can define f in the interval $[b, b']$ in the same way as in Lemma 11 such that it becomes a bounded function of class C^1 and all hypotheses of Theorem 2 hold for the extended system f, α, β, ϕ, ψ over the interval $[a, b']$.

Step 2. We extend $\alpha, \beta, \phi, \psi, f$ into the range $t > b'$ as follows.

$$f = -x, \qquad \beta = 0, \qquad \alpha = c \sin(t - b' + \kappa), \tag{4.14}$$

$$\phi(t, x) = \phi(b', x) = -\sqrt{\phi_0}\,, \qquad \psi(t, x) = \psi(b', x) = \sqrt{\psi_0}\,,$$

where the constants c, κ have to be chosen such that α and α' are continuous at $t = b'$. This can always be done with $c < 0$, $0 < \kappa < \pi/2$ since $\alpha(b') < 0$, $\alpha'(b') < 0$ in view of (4.9). Therefore we can find a number $b'' > b'$ such that we have

$$b'' - b' > \pi/2, \qquad \alpha < 0 \qquad \text{for all} \qquad t \in [b', b'']. \tag{4.15}$$

It is clear that α is a lower, β an upper solution of the extended differential equation on the whole interval $[a, b'']$ and that $\alpha < \beta$ there. Furthermore, we have from (4.13) and (4.14) that

$$\alpha^2 + (\alpha')^2 = \text{const.} = r^2$$

and hence

$$\begin{gathered} |\alpha| \leqslant r, \qquad |\alpha'| \leqslant r, \qquad \phi(t, \alpha) < \alpha' < \psi(t, \alpha), \\ \phi(t, \beta) < \beta' < \psi(t, \beta) \end{gathered}$$

for all $t \geqslant b'$.

Finally we observe that for $t > b'$ the expressions

$$f(t, x, \phi) - \phi_x \phi - \phi_t \qquad \text{and} \qquad f(t, x, \psi) - \psi_x \psi - \psi_t$$

have the form

$$-x - \tfrac{1}{2}(\phi_0)_x \qquad \text{and} \qquad -x - \tfrac{1}{2}(\psi_0)_x$$

respectively. So they are $\neq 0$ and have the same sign as in the range $t \leqslant b'$, according to (4.11), (iii).

As a result of the two extension procedures we have obtained a system $\alpha, \beta, \phi, \psi, f$, which reduces to the given one if $t \in [a, b]$, satisfies all conditions of Theorem 3 with respect to the interval $[a, b'']$ and which has the following property:

$$\text{There exists a } b' \text{ with} \quad b \leqslant b' \leqslant b'' \quad \text{such that}$$
$$b'' - b' > \pi/2 \quad \text{and} \quad f = -x \quad \text{if} \quad b' < t < b''. \tag{4.16}$$

(See (4.14), (4.15).)

We then extend the definition of $\alpha, \beta, \phi, \psi, f$ in a similar way once more, but now into the range $t < a$, with this result: All hypotheses of Theorem 3 are satisfied over some interval $[a'', b'']$, $a'' < a$, everything reduces to the original data if $t \in [a, b]$, (4.16) holds and also the analogue:

$$\text{There exists an } a' \text{ with} \quad a'' \leqslant a' < a \quad \text{such that}$$
$$a' - a'' > \pi/2 \quad \text{and} \quad f = -x \quad \text{if} \quad a'' < t < a'. \tag{4.16'}$$

To establish (4.16) and (4.16′) was the purpose of the preceding considerations. How we are going to make use of them will become clear from

LEMMA 12. *Let ξ_1, ξ_2 be two solutions of the extended differential equation such that $\Delta = \xi_1 - \xi_2$ does not change sign on $[a'', b'']$. Then Δ' has a zero in each of the intervals $[a'', a']$, $[b', b'']$ and we have*

$$\operatorname*{Min}_{t \in [a,b]} |\Delta| \geqslant \operatorname*{Min}_{t \in Z} |\Delta|,$$

where

$$Z = \{t : t \in [a'', b''], \Delta'(t) = 0\}.$$

Proof. ξ_1 and ξ_2 are solutions of $x'' = -x$ on the subintervals (a'', a'), (b', b''). The same is then true for their difference, and therefore Δ can be written in the form $k_1 \sin(t + k_2)$ for each of the two intervals. If the function $\sin t$ has no zero in an interval of length $> \pi/2$ its derivative certainly has one. So the first statement of the lemma follows immediately from (4.16), (4.16′), the second statement follows then from the first one and from Lemma 7 (we may assume without loss of generality that $\Delta \geqslant 0$ on I, so that $\operatorname{Min} |\Delta| = \operatorname{Min} \Delta$).

We are now ready to apply the results of Lemma 11 and its corollary to the extended differential equation. As we have already observed at the beginning of this proof there exists a solution ξ over $I_2 = [a'', b'']$ which satisfies (4.8) and which can be approximated by a sequence of solutions ξ_ν such that (1.2) (i), (ii) hold for all $t \in I_2$. Using Lemma 6 we will now show that this sequence ξ_ν has also property (1.2) (iii) at least on the subinterval $[a, b]$ (and that is all what we want to know).

To this purpose we first have to extend the definition of f into the region $I_2 \times R^2$. We do this in the following way

$$f(t, x, x') = \begin{cases} f(t, \alpha(t), x') & \text{for} \quad x < \alpha(t), \\ f(t, \beta(t), x') & \text{for} \quad x > \beta(t), \end{cases}$$

which guarantees that $f \in (C_t, \text{lip})$ on the full space. Now all trajectories (t, ξ_ν, ξ_ν') are contained in a certain bounded subregion D of $I_2 \times R^2$ and $|\Delta_\nu|$, $|\Delta_\nu'|$ are bounded by a constant δ which is independent of ν and t. As we have seen in Lemma 6, there are constants k_0, k_1 (depending on D, δ, f only) such that for any pair of numbers $t, u \in I_2$ the inequality

$$|\Delta_\nu'(t)| \leqslant k_0 |\Delta_\nu(u)| + k_1 |\Delta_\nu'(u)| \tag{4.17}$$

holds. Consider now the set

$$Z_\nu = \{t : t \in I_2, \Delta_\nu'(t) = 0\}.$$

Z_ν is closed and not empty according to Lemma 12. It follows then from (4.17) that

$$|\Delta_\nu'(t)| \leqslant k_0 \operatorname*{Min}_{u \in Z_\nu} |\Delta_\nu(u)|$$

for all $t \in I''$. Applying Lemma 12 again we obtain

$$|\Delta\ '(t)| \leqslant k_0 \operatorname*{Min}_{u \in [a,b]} |\Delta_\nu(u)|$$

and therefore

$$|\Delta_\nu'(t)| \leqslant k_0 |\Delta_\nu(t)|$$

if $t \in [a, b]$. Hence the theorem is proved.

5. Applications

We will now discuss some simple consequences of the foregoing results. Henceforth all functions f, g, h etc. are supposed to be of class C_t and have derivatives $f_x = (\partial/\partial x) f$, $f_{x'} = (\partial/\partial x') f$ also of class C_t in some sufficiently large region of $I \times R^2$.

The following lemma provides a method to compare solutions of nonlinear equations with those of linear ones.

LEMMA 13. *Given two differential equations*

$$x'' = f_1(t, x, x') \tag{5.1}$$

$$x'' = f_2(t, x, x'), \tag{5.2}$$

where f_2 is linear

$$f_2 = \lambda(t)\, x' + \mu(t)\, x + \gamma(t)$$

and such that the homogeneous equation

$$x'' = \lambda x' + \mu x \tag{5.3}$$

has a nontrivial solution with more than one zero in the interior of I. Let there be given also a solution ξ_1 of (5.1) and a solution ξ_2 of (5.2) with $\xi_2 < \xi_1$ for all $t \in I$.

Let now $h = h(t, x)$ be a function of t and x such that

$$h_x \leqslant \mu(t) \tag{5.4}$$

for all (t, x) with $\xi_1 \leqslant x \leqslant \xi_2$. Then the trajectory (t, ξ_1, ξ_1') cannot be in the region

$$f_1(t, x, x') - h(t, x) - \lambda(t)\, x' - \gamma(t) + h(t, \xi_2) - \mu(t)\, \xi_2 \leqslant 0 \tag{5.5}$$

for all $t \in I$.

Proof. Put $g = f - h$. Assume that we have

$$g(t, \xi_1, \xi_1') - \lambda\xi_1' - \gamma + h(t, \xi_2) - \mu\xi_2 \leqslant 0 \tag{5.6}$$

for all $t \in I$, contrary to the statement of the lemma.

Consider now the function

$$f(t, x, x') = \lambda(t)\, x' + h(t, x) + g(t, \xi_1, \xi_1') - \lambda(t)\, \xi_1'.$$

Obviously

$$f(t, \xi_1, \xi_1') = f_1(t, \xi_1, \xi_1')$$

and this means that ξ_1 is a solution of

$$x'' = f(t, x, x'). \tag{5.7}$$

Since ξ_2 is a solution of (5.2) we have

$$\begin{aligned} -\xi_2'' + f_1(t, \xi_2, \xi_2') &= -\xi_2'' + \lambda\xi_2' + h(t, \xi_2) + g(t, \xi_1, \xi_1') - \lambda\xi_1' \\ &= -\mu\xi_2 - \gamma + h(t, \xi_2) + g(t, \xi_1, \xi_1') - \lambda\xi_1' \leqslant 0 \end{aligned}$$

from (5.6). So $\xi_2 < \xi_1$ is a lower solution of (5.7). Furthermore f is linear in x', so one can easily find functions ϕ, ψ with all properties required in Theorem 3 (see [7], Lemma 6, p. 187). Therefore the differential equation (5.7) has a solution ξ with property (B) and $\xi_1 \leqslant \xi \leqslant \xi_2$. On the other hand $f_{x'} = \lambda(t)$ and $f_x = h_x \leqslant \mu$ for $x = \xi$, $x' = \xi'$ and $t \in I$ because of (5.4). But then, by Theorem 1, every solution of (5.3) has at most one zero in I and this contradicts the hypotheses of the lemma.

We now apply the lemma to an autonomous equation of the form

$$x'' = g(x, x') - x. \tag{5.8}$$

THEOREM 4. *Let ξ be a solution of (5.8) with $\xi(t_0) > 0$ for a certain t_0. Put $\vartheta_0 = \arg(\xi(t_0), \xi'(t_0))$ with $|\vartheta_0| < \pi/2$. Let there be given a number $\gamma > 0$. Then the curve (ξ, ξ') reaches either the half-plane $x \leqslant \gamma$ or the region $g(x, x') \geqslant \gamma$ for some t with $t_0 \leqslant t \leqslant t_0 + \vartheta_0 + \pi/2$.*

Proof. Assume we have $\xi > \gamma$ for all $t \in [t_0, t_0 + \vartheta_0 + \pi/2]$. Then there exists a $b > t_0 + \vartheta_0 + \pi/2$ such that this inequality holds for all $t \in [t_0, b]$. We choose now a number a with

$$t_0 + \vartheta_0 - \pi/2 < a < t_0, \qquad b - a > \pi \tag{5.9}$$

and define two functions f_1, f_2 as follows.

$$f_1 = \begin{cases} -x & \text{for} \quad a \leqslant t \leqslant t_0 \\ g(x, x') - x & \text{for} \quad t_0 < t \leqslant b, \end{cases}$$

$$f_2 = \begin{cases} -x & \text{for} \quad a \leqslant t \leqslant t_0 \\ -x + \gamma & \text{for} \quad t_0 < t \leqslant b. \end{cases}$$

Let ξ_1 be the solution of $x'' = f_1(t, x, x')$ with

$$\xi_1(t_0) = \xi(t_0), \qquad \xi_1'(t_0) = \xi'(t_0),$$

and ξ_2 the solution of $x'' = f_2(t, x, x')$ with $\xi_2(t_0) = 0$, $\xi_2'(t_0) = 0$. We have then

$$\xi_2 = \begin{cases} 0 & \text{for} \quad a \leqslant t \leqslant t_0 \\ \gamma(1 - \sin(t - t_0 + \pi/2)) \leqslant \gamma & \text{for} \quad t_0 < t \leqslant b, \end{cases}$$

$$\xi_1 = \begin{cases} c\cos(t - t_0 - \vartheta_0) > 0, \quad c > 0, & \text{for} \quad a \leqslant t \leqslant t_0 \\ \xi & \text{for} \quad t_0 < t \leqslant b. \end{cases}$$

Hence $\xi_2 < \xi_1$ for all $t \in [a, b]$ because of the assumption $\xi < \gamma$.

Now all hypotheses of Lemma 13 are satisfied if we put $h = -x$, $\mu = -1$, $\lambda = 0$, $\gamma(t) = 0$ for $t \leqslant t_0$ and $\gamma(t) = \gamma$ for $t > t_0$. We have then

$$f_1 - h = \begin{cases} 0 & \text{for} \quad a \leqslant t \leqslant t_0 \\ g(x, x') & \text{for} \quad t_0 < t \leqslant b, \end{cases}$$

and the left hand side of (5.5) reduces in this case to the expression $f_1 - h - \gamma$ with $\gamma > 0$. So if the inequality (5.5) does not hold along the solution ξ, the violation can only happen in the range $t \geqslant t_0$. Hence $g(\xi, \xi') > \gamma$ for some t_1 with $t_0 \leqslant t_1 \leqslant b$. Since b can be chosen as close to $t_0 + \vartheta_0 + \pi/2$ as one wishes we must have $g(\xi, \xi') \geqslant \gamma$ somewhere in $[t_0, t_0 + \vartheta_0 + \pi/2]$. Thus the theorem is proved.

Example. Let us consider the autonomous equation of the van der Pol-type

$$y' = k(x)\, y - x, \qquad y = x', \tag{5.10}$$

where k is of class C^1 for $x \geqslant 1$ and satisfies the conditions

$$k(1) = 0, \qquad k_x < 0, \qquad (x/k)_x > 0 \qquad \text{for} \qquad x > 1. \tag{5.11}$$

(All this is of course true for $k = c(1 - x^2)$.) It follows from (5.10) that $k < 0$ and

$$k(x/k) - x - (x/k)_x\,(x/k) = -\,(x/k)_x\,(x/k) > 0$$

for $x > 1$. Therefore the solution curves cross the surface $y = \theta = x/k$ always upwards (see (2.11)) and a trajectory (ξ, ξ'), moving down from the x-axis has to stay above $y = \theta$:

$$\xi' > \frac{\xi}{k(\xi)} \tag{5.12}$$

(see Fig. 1).

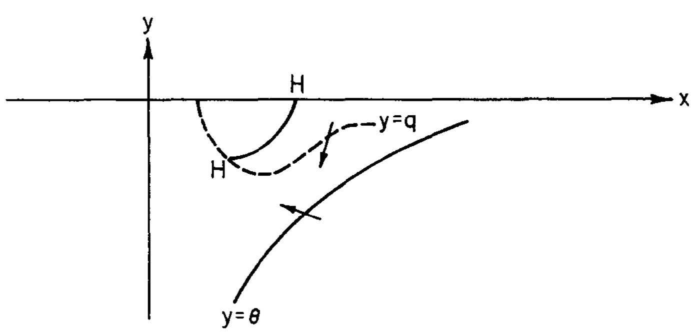

FIG. 1

We are now going to show that the negative root $y = q(x)$ of the quadratic equation

$$y\frac{k_x}{k} - \frac{x}{y} + k = 0 \tag{5.13}$$

plays a role which is somehow opposite to the one of $y = \theta$: The curve can be reached from every point of the x-axis in time $\leqslant \pi/2$. Furthermore, if we require k to satisfy two additional conditions, a trajectory can never cross $y = q$ upwards and (ξ, ξ') remains between $y = q$ and $y = \theta$ nearly the whole way from the x-axis to the line $x = 1$. This, of course, gives some information about the location of the trajectory. How good this information is may be judged from the inequalities

$$\frac{x}{k} - 2\left(\frac{x}{k}\right)^2 \frac{k_x}{k^2} > q > \frac{x}{k}$$

which can be easily obtained with the use of Taylor's formula from the explicit representation of q, that is,

$$\begin{aligned} q &= -\frac{k^2}{2k_x} - \sqrt{\left(\frac{k^2}{2k_x}\right)^2 + x\frac{k}{k_x}} \\ &= -\frac{k^2}{2k_x} + \frac{k^2}{2k_x}\sqrt{1 + 4\left(\frac{x}{k}\right)\frac{k_x}{k^2}}\,. \end{aligned} \tag{5.14}$$

Equation (5.13) allows a simple geometric interpretation. Let us consider the family of curves defined by

$$y = \gamma/k \quad (\gamma \text{ a real parameter}).$$

The set of all points in the phase-plane, where a member of the family touches an orbit, is then identical with the locus of Eq. (5.13). Indeed, these points are determined by the two equations

$$y = \frac{\gamma}{k}\,, \qquad -\frac{x}{y} + k = -\frac{\gamma}{k^2}\,k_x\,.$$

Elimination of γ leads to (5.13).

Since $q \to 0$ for $x \to 1$ (see (5.14)) it is clear that a trajectory (ξ, ξ') coming from a point H on the x-axis, has to cross $y = q$ at a certain point H^* before reaching $x = 1$ (see Fig. 1). We are now going to show that (ξ, ξ') always runs through the arc HH^* in time $\leqslant \pi/2$. Let us assume that (ξ, ξ') starts in H at time $t = 0$, that is, $\xi(0) = h > 1$, $\xi'(0) = 0$. If $\xi(\pi/2) \leqslant 1$ nothing has to be proved. So we may assume $\xi(\pi/2) > 1$. Take now $\epsilon > 0$ and sufficiently small and put $\gamma = \xi(\pi/2 + \epsilon)$. Since $\xi' \leqslant 0$ we have then $\xi > \gamma$ for all $t \in [0, \pi/2]$. We then apply Theorem 4 with $g = k(x)\,x'$, $\vartheta_0 = 0$, and $t_0 = 0$:

There is a $t_1 \in [0, \pi/2]$ such that $k(\xi)\,\xi' \geqslant \gamma$ or

$$\xi' \leqslant \frac{\gamma}{k(\xi)} \qquad \text{for} \qquad t = t_1\,.$$

On the other hand we have

$$0 = \xi' > \frac{\gamma}{k(\xi)} \qquad \text{for} \qquad t = 0$$

$$\xi' > \frac{\gamma}{k(\xi)} = \frac{\xi}{k(\xi)} \qquad \text{for} \qquad t = \frac{\pi}{2} + \epsilon,$$

because of (5.12).

So we have the situation indicated in Fig. 2, where P_0, P_1, P_2 refer to the positions of (ξ, ξ') at the times $t = 0$, t_1, $\pi/2 + \epsilon$ respectively. It is then obvious that there exists a curve of the form $y = \gamma/k$ with $\gamma \geqslant \xi(\pi/2 + \epsilon)$ which touches the orbit somewhere between P_0 and P_2. So (ξ, ξ') reaches $y = q$ at a certain time $\leqslant \pi/2 + \epsilon$ and therefore, since $\epsilon > 0$ can be arbitrarily small, also at a time $\leqslant \pi/2$.

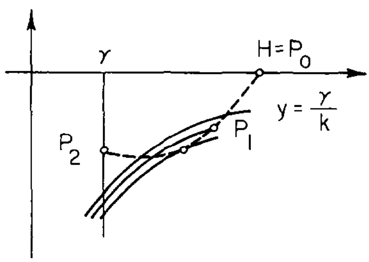

FIG. 2

This result leads immediately to an estimate of the x-coordinate h^* of H^* in terms of h. We have in view of (5.12)

$$\frac{\pi}{2} \geqslant \int_{h^*}^{h} \left| \frac{k}{x} \right| dx.$$

So a condition like

$$\int_1^{\infty} \left| \frac{k}{x} \right| dx = \infty \tag{5.15}$$

will certainly guarantee that $h^* - h \to 0$ for $h \to \infty$. It is clear from Fig. 1 that $h^* - h \to 0$ for $h \to 1$. Now if the mapping $h \to h^*$ (or what amounts to the same, the mapping $H \to H^*$) is continuous, every point of $y = q$ has to be an H^*, or what is the same, trajectories can cross $y = q$ only downwards. It can be shown by a straightforward consideration, which is out of place here, that $H \to H^*$ will always be a continuous mapping, if the curve $y = 1/k$ is convex from above for all $x > 1$.

REFERENCES

1. BIEBERBACH, L. "Einführung in die Theorie der Differentialgleichungen im reellen Gebiet." Springer, Berlin, 1956.
2. CARTWRIGHT, M. L. Van der Pol's equation for relaxation oscillations. In "Contributions to the Theory of Nonlinear Oscillations," Vol. 2, pp. 3-18. Princeton Univ. Press, Princeton, New Jersey,
3. CESARI, L. "Asymptotic Behaviour and Stability Problems in Differential Equations." Ergebnisse der Mathematik und ihrer Grenzgebiete N.F., Vol. 16, 2nd ed. Springer, Berlin, 1963.
4. CESARI, L. Functional analysis and periodic solutions of nonlinear differential equations. *Contrib. Diff. Eqs.* **1**, 149-187 (1963).
5. CODDINGTON, E. A., AND LEVINSON, N. "Theory of Ordinary Differential Equations." McGraw Hill, New York, 1955.
6. KNOBLOCH, H. W. Remarks on a paper of L. Cesari on functional analysis and nonlinear differential equations. *Mich. J. Math.* **10**, 417-430 (1963).
7. KNOBLOCH, H. W. Eine neue Methode zur Approximation periodischer Lösungen nicht-linearer Differentialgleichungen zweiter Ordnung. *Math. Z.* **82**, 177-197 (1963).